12.4 Computational appendices

12.4.1 Derivation of the Rayleigh Jean spectrum and Planck's law

In order to get the energy density of a radiation of frequency ν emitted by a blackbody heated at temperature T, noted $u(\nu, T)$, we start by a very common calculation at the end of the 19th century:

$$u(\nu, T) = N_{mfv}(\nu)\bar{\epsilon}_m(\nu, T) \tag{103}$$

Where N_{mfv} is the number of possible modes the radiation can have at frequency ν per volume units, and $\bar{\epsilon}_m$ is the average energy per modes. To simplify, we will first assume the blackbody to be a perfect cavity.

To assess the number of modes the radiation of frequency ν can have per unit of volume in the cavity, we'll start from Maxwell's equation for the propagation of light (wave equation, see Eq. 44):

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\overrightarrow{E}(x, y, z, t) = \overrightarrow{0} \tag{104}$$

This is a linear homogeneous equation. We can thus search for eigenmodes by decomposing the electric field as $E(x, y, z, t) = X(x)Y(y)Z(z)T(t)$ (see Sturn-Liouville theory). This allows to rewrite the wave equation as

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} \tag{105}$$

The left hand side of the equation only depends on time and must be equal to a constant, called $-\omega^2$, and same for the spatial terms. In addition this constant must be negative because we are looking for an oscillating solution, and not an exponentially decaying one. Therefore:

$$\begin{cases} \frac{T''(t)}{T(t)} & = -\omega^2 \\ \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} & = -\omega^2 c^{-2} \end{cases} \tag{106}$$

We can split again the spatial differential equation as:

$$\begin{cases} X''(x) & = -k_x^2 X(x) \\ Y''(y) & = -k_y^2 Y(y) \\ Z''(z) & = -k_z^2 Z(z) \end{cases} \tag{107}$$

With $k^2 = \omega^2 c^{-2} = k_x^2 + k_y^2 + k_z^2$. The solutions of these set of 4 ordinary differential equations are:

$$\begin{cases} X(x) & = A_x sin(k_x x) \\ Y(y) & = A_y sin(k_y y) \\ Z(z) & = A_z sin(k_z z) \\ T(t) & = A_t sin(\omega t) \end{cases} \tag{108}$$

If we now consider the blackbody as a cubic cavity of length L, by imposing vanishing fields on the edges of the cavity we get an expression for the wavevector:

$$k_x = \frac{n_x \pi}{L} \qquad k_y = \frac{n_y \pi}{L} \qquad k_z = \frac{n_z \pi}{L} \tag{109}$$

We can then reconstruct the resulting electric field as:

$$E(x, y, z, t) = E_0 sin(\frac{n_x \pi x}{L})sin(\frac{n_y \pi y}{L})sin(\frac{n_z \pi z}{L})sin(\omega t) \tag{110}$$

With $\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2)$, we can get the following expression for the wavelength $\lambda = \nu c$:

$$n_x^2 + n_y^2 + n_z^2 = \frac{4L^2}{\lambda^2} \tag{111}$$

This equation provides a way to know the number of modes associated to a wave of frequency ω in the cavity. This is simply the number of combination of positive (n_x, n_y, n_z) leading to the same value of $n_x^2 + n_y^2 + n_z^2$, including degeneracies due to polarization. To do this simply, note that Eq. 111 is a sphere equation of radius $n = \frac{2L}{\lambda} = \frac{2L\nu}{c}$. The number of modes $N(\nu)$ between frequencies ν and $\nu + d\nu$ is given by

$$N(\nu)d\nu = \frac{1}{8} \times 2 \times 4\pi n^2 dn = \pi n^2 dn = \frac{8\pi}{c^3} L^3 \nu^2 d\nu \tag{112}$$

In the equation above, we differentiated the sphere of radius n. $4\pi n^2 dn$ is the 3d volumic infinitesimal element. The 2 factor account for polarization: for a given mode, there is two polarization directions possible. Then, the factor $\frac{1}{8}$ is here to account only for positive values of n, meaning the one-eigth of the total sphere where all values of n_x, n_y and n_z are positive.

NB: if we do not want to consider a cavity (for instance just the emission of heated quartz), in the limit of $L \to \infty$ the number of modes remain the same (simply replace the discrete sum over n_x, n_y, n_z by an integral). The cavity argument is a convenient way to discretize the modes.

Finally, we get the expression of the density of modes at frequency ν,

$$N_{mfv}(\nu) = \frac{8\pi\nu^2}{c^3} \tag{113}$$

Now to determine the average energy per node, this is where Planck hypothesis changes everything. In the fully classical computation done by Rayleigh and Jeans, we use the Boltzmann distribution typical in statistical physics and get:

$$\bar{\epsilon}_m(\nu, T) = \frac{\int_0^\infty \epsilon e^{-\beta\epsilon} d\epsilon}{\int_0^\infty e^{-\beta\epsilon}} = \frac{1}{\beta} = k_B T \tag{114}$$

This expression would lead to the Rayleigh-Jean spectrum when multiplied by the density of modes, which, as explained in the main text, leads to the UV catastrophy as this diverges when $\nu \to \infty$.

We now introduce the quantum assumption made by Planck:

$$\boxed{\epsilon_m^l = lh\nu} \qquad \textbf{PLANCK'S QUANTIZATION} \tag{115}$$

where h is the Planck constant and l is an integer. Under this assumption, if we compute the average energy per mode using Boltzmann distribution, we get:

$$\bar{\epsilon}_m^l = \frac{\sum_{l=0}^{\infty} lh\nu e^{-l\beta h\nu}}{\sum_{l=0}^{\infty} e^{-l\beta h\nu}} \tag{116}$$

These series can be computed analytically: we can recognize geometrical expansions:

$$\begin{cases} \sum_{l=0}^{\infty} e^{-l\beta h\nu} & = \sum_{l=0}^{\infty} (e^{-\beta h\nu})^l = \frac{1}{1-e^{-\beta h\nu}} \\ \sum_{l=0}^{\infty} lh\nu e^{-l\beta h\nu} & = -\frac{1}{\beta} \frac{d}{d(h\nu\beta)} \sum_{l=0}^{\infty} e^{-lh\nu\beta} = \frac{h\nu e^{-\beta h\nu}}{(1-e^{-\beta h\nu})^2} \end{cases} \tag{117}$$

Thus, using Planck's quantization we get the following expression of the average energy per mode:

$$\bar{\epsilon}_m = \frac{h\nu}{e^{\beta h\nu} - 1} \tag{118}$$

This last expression, when multiplied by the density of modes of frequency ν, leads to the expression of Planck's law provided in the main text Eq. 75.

Inspiration material: [6]