

4 Classical mechanics

4.1 Newtonian point mechanics

The commonly accepted root of what we now call modern physics is mechanics. As science is always a matter of sharing knowledge and building from what other discovered, it is a bit presumptuous to state that Newton was the father of point mechanics. Nevertheless, he is definitely remembered as such.

4.1.1 Frames

The core concept of mechanics is simple: describe mathematically the motions of objects in space and time. Such a description requires a way of measuring motion: we need a way to associate local direction at each point of space and a time parameter. A frame is defined as a choice of directions at each points of space and a time parameter relative to which motion is described.

But how exactly can we describe this mathematically ? This type of problems are typically treated in a branch of mathematics called "differential geometry". The core idea is to describe space as a structure called a manifold. This description allows to assign a set of basis vectors forming a vector space called "tangent space" at each point of the manifold (the physical space). The vectors belonging to these tangent spaces can represent velocities of particles moving through space. While a vector field assigns one tangent vector to each point of the manifold, assigning a basis of the tangent space at each point of the manifold is what mathematicians call a frame. The interested reader can refer to the appendix on topology to learn more.

However, to build a frame to study motion in physics, we need a bit more than just a velocity vector at each point of space. It also needs a time parameter with respect to which velocities vary smoothly. In addition, an origin of space and a time parameter is required to express coordinates. In essence, this is what a frame really is: In classical mechanics, space is usually modeled as \mathbb{R}^3 , which allows us to choose an origin and define position vectors. The frame in its mathematical sense associate a basis of tangent vector to all points of \mathbb{R}^3 . Then we introduce a time parameter defined on an interval, and assume tangent vectors vary smoothly with respect to it.

Then, we can use frames to represent any states of motion. But it is also very convenient to be able to label points. This is where we introduce coordinate systems. Most if not all frames in physics use them (the frame is then called holonomic). However, it is worth noting that some few physically important frames (e.g. rotating frames) are anholonomic. A coordinate system is a bijective map from the manifold to \mathbb{R}^3 that assigns three numbers to each point. The derivatives of the coordinate functions induce basis vectors of the tangent space. Therefore we often say a coordinate system induces a frame via derivative (provided an origin of space and time, and a time parameter).

Below, we list the three most used coordinate systems and represent the associated frames.

We can explicitly write a few coordinate systems usual in physics:

$$\begin{cases} \vec{r} &= x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \\ \vec{r} &= r(\cos(\theta)\vec{e}_x + \sin(\theta)\vec{e}_y) + z\vec{e}_z \\ \vec{r} &= r(\sin(\theta)\cos(\phi)\vec{e}_x + \sin(\theta)\sin(\phi)\vec{e}_y + \cos(\theta)\vec{e}_z) \end{cases} \quad (1)$$

The coordinates systems above are respectively called Cartesian, cylindrical and spherical coordinates. They induce a frame, because they naturally provide a basis to decompose velocity vectors.

Among all possible frames, some are special: in them, free motion is simple. These are called Galilean (inertial) frames. In these frames, any point on which no forces are exerted or on which the resultant of the forces is zero is in uniform rectilinear motion, or at rest.

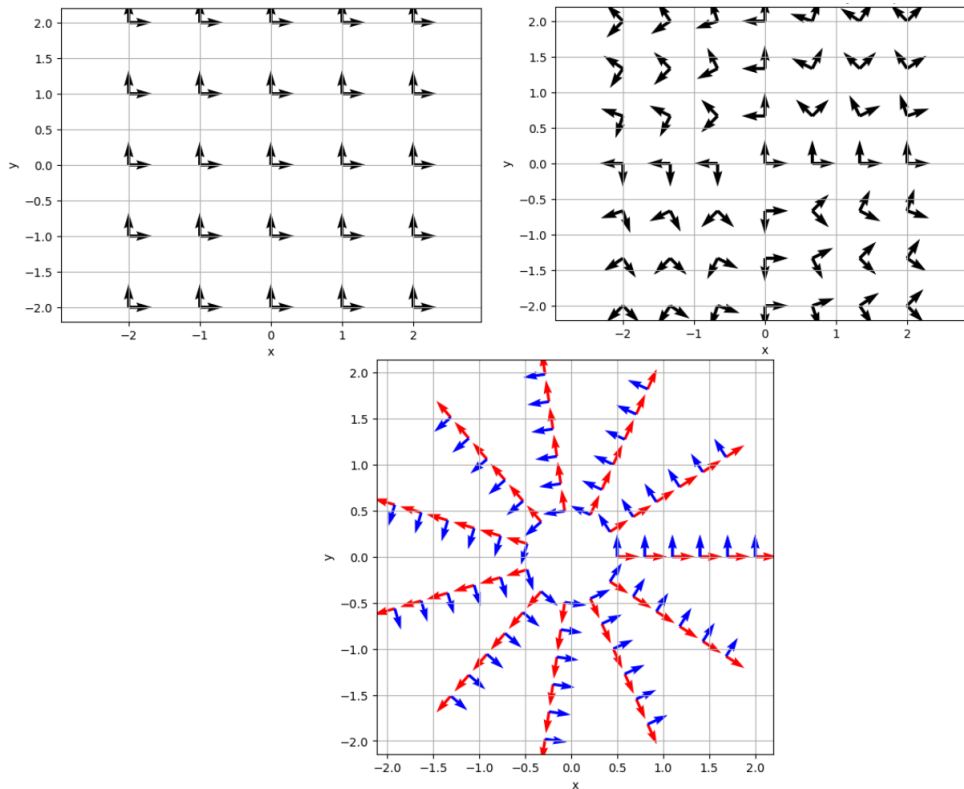


Figure 1: Examples of a Cartesian, polar and an anholonomic frame respectively. The two first frames are induced by a coordinate system, but not the third one (basis vectors of the tangent space do not seem to be obtained from derivatives of coordinates). It is crucial to understand that these vectors form a basis of the tangent space at a point of the manifold. The vectors written using these basis are tangent space vectors, for instance velocities. On the other hand, we can define position vectors as those connecting an arbitrary origin to a point.

To summarize, there's a hierarchy on how to build mathematically an inertial frame. First, we need to describe space by a manifold (usually \mathbb{R}^3 in classical mechanics which is a vector space so it can have an associated origin). Then, as we mostly work with holonomic frames, we define coordinates system that will induce a basis and a general frame (in the mathematical sense). To get a reference frame (in the physical sense), we add a time parameter and an origin. Finally, an inertial frame is a referential frame choice for a given system so that the motion is uniform rectilinear in the absence of applied forces or at equilibrium.

4.1.2 Newton's momentum, Leibniz's energy and the meaning of conservation

We can now state the first approach to study mechanical systems by introducing Newton's laws. Historically, the laws were an attempt to describe both the free fall equations and the motion of celestial bodies within a unified frame. As the mathematical tools didn't exist yet, Newton even sketched the first ideas of what would later be called differential geometries. The usual modern way to introduce Newton laws is the following:

- The existence of Galilean frame (Newton 1st law)
- Principle of fundamental dynamics (Newton 2nd law):

$$m \vec{a} = \sum_s \vec{F}_s \quad (2)$$

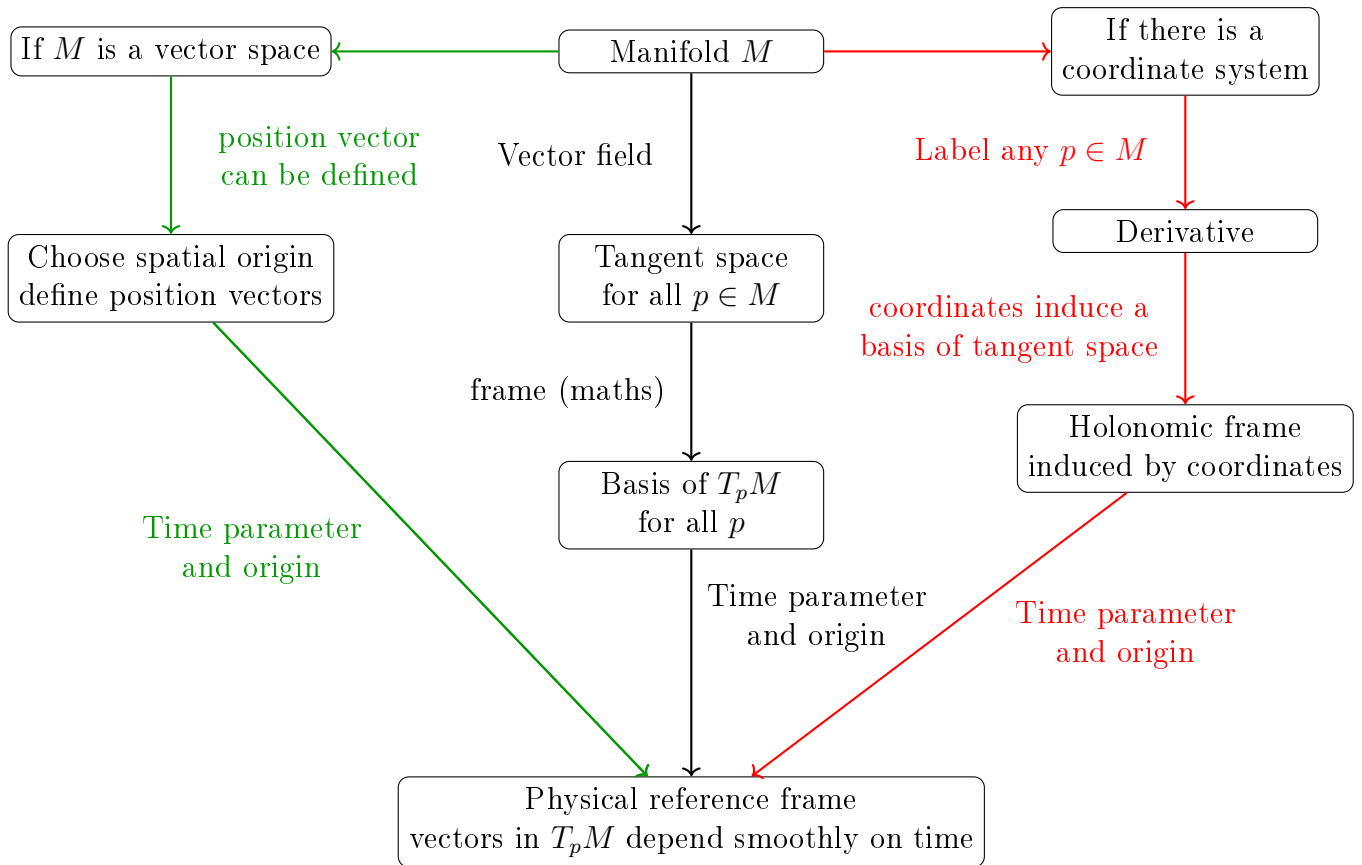


Figure 2: Sketch of how frames are built from mathematics in order to be used in physics. Note that the artificial addition of the time parameter disappear when considering special relativity, as time becomes a dimension of the manifold.

Where m is the mass of the system assimilated to a point, \vec{a} the system acceleration and \vec{F}_S the external forces applied to the point-like system.

- Finally, Newton's third law states that interactions between two bodies occur in equal and opposite pairs: $\vec{F}_{AB} = -\vec{F}_{BA}$.

A subtle point often glossed over is why the basic mechanical quantities have the forms we now take for granted in inertial frames. Momentum for instance, is not an arbitrary construct. Indeed, the symmetry of interactions suggested the existence of a quantity exchanged between bodies in collisions. Experiment shows that this quantity \vec{p} must be proportional to velocity and the constant of proportionality is what we now call "a mass". Once momentum is defined, Newton's second law becomes simply its time evolution:

$$\dot{\vec{p}} = \sum_{i \in \text{ext.}} \vec{F}_i \quad \text{with} \quad \boxed{\vec{p} = m \vec{v}} \quad \text{NEWTON'S 2ND LAW} \quad (3)$$

In parallel to Newton's work, Leibniz introduced by studying elastic collision another quantity, that he assumed to be "conserved": he called it "vis viva" mv^2 .

Both Newton with his momentum and Leibniz with his vis viva were convinced that their physical quantity were "conserved". In reality, both were correct to some extent. In the XVIIIth century, Smeaton and Bernoulli empirically observed and confirmed through their engineering projects that a quantity they called "work" W seemed to be destroying or creating Leibniz's vis viva:

$$W_{A \rightarrow B} = \int_A^B \vec{F} \cdot d\vec{r}$$

WORK

(4)

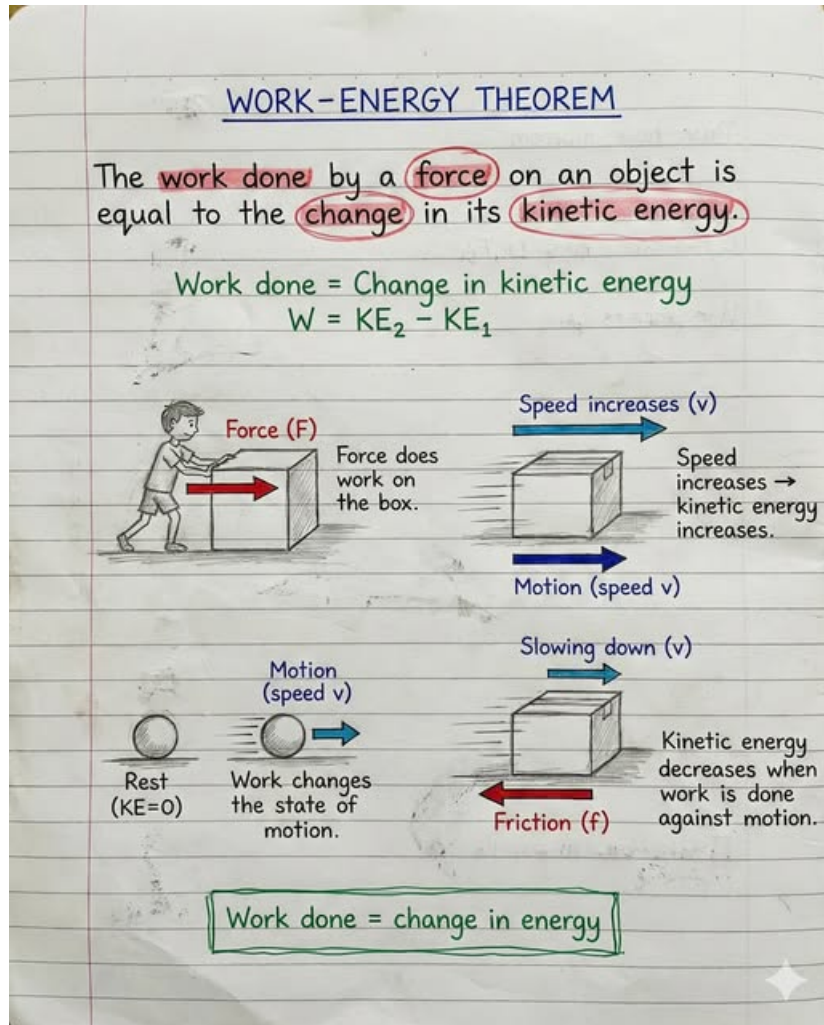


Figure 3: illustration on the work-energy theorem. Credit: edusphere academy.

Where \vec{F} is a considered force and \vec{r} the considered displacement vector for the system. It turns out work really was the quantity that helped understanding under which circumstances momentum or energy is conserved. Indeed, let's consider the fluctuations of the work on a path with time, that is, its time derivative:

$$\frac{dW}{dt} = \vec{F} \cdot \vec{v} \quad (5)$$

Using Newton's 2nd law, one gets $\dot{W} = \dot{\vec{p}} \cdot \vec{v} = m \dot{\vec{v}} \cdot \vec{v}$. That expression can be rewritten by considering the following elegant differential identity:

$$\frac{dW}{dt} = m \frac{d\vec{v}}{dt} \cdot \vec{v} = \frac{d}{dt} \left(\frac{1}{2} m v^2 \right) \quad (6)$$

this result shows that the work done by forces between two instants depends only on the variation of the quantity $\frac{1}{2} m v^2$. This suggest introducing a new state function, called the kinetic energy E_c :

$$\boxed{E_c = \frac{1}{2}mv^2} \quad \text{KINETIC ENERGY} \quad (7)$$

Work is then defined as a kinetic energy difference: $W_{t_0 \rightarrow t} = E_k(t) - E_k(t_0)$, it depends on the process (path, force applied..). In a more general manner, we could write that work represents the energy from state A to state B , while kinetic energy is a quantity that can be defined for both states

$$\boxed{W_{A \rightarrow B} = E_k(B) - E_k(A)} \quad \text{WORK-ENERGY THEOREM} \quad (8)$$

It is interesting to notice the $\frac{1}{2}$ factor difference with Leibniz original "vis viva". It is not mv^2 which is conserved, but a quantity whose variation is controlled by work which naturally introduced that $\frac{1}{2}$ factor. **Work is not energy, it is the mechanism by which energy changes. It represents an energy transfer.**

- **Force:** instantaneous, acts locally in time.
- **Work:** global, accumulate over a trajectory
- **kinetic energy:** state functions whose variation equals to work

What does conservation means in point mechanics ?

Now we have everything we need to really ask the question: what is conserved, and under what assumptions ? First, a rigorous mathematical definition. We note \mathcal{S} the "state space": in point mechanics, it is defined for a particle by its position and velocity (or equivalently by position and momentum):

$$\mathcal{S}_{mecha} = \{(x, \dot{x})\} \quad \text{STATES SPACE} \quad (9)$$

It is crucial to not confuse state space with the trajectories. A state s is just a point, defined at one instant t_0 :

$$s \in \mathcal{S} \Rightarrow \exists t_0 \in \mathbb{R} / s = (x(t_0), \dot{x}(t_0)) \quad \text{ONE STATE} \quad (10)$$

On the other hand, a trajectory is just a curve in state space. It is a map γ such that

$$\begin{aligned} \gamma : \mathbb{R} &\rightarrow \mathcal{S} \\ t &\mapsto (x(t), \dot{x}(t)) \end{aligned} \quad \text{A TRAJECTORY} \quad (11)$$

Among all trajectories, some are physically allowed if they "solve the equations of motions". This is equivalent to be a solution of Newton's 2nd law (Eq. 3). More precisely, with our definitions, this means $\gamma(t)$ is solution of

$$\partial_t \gamma(t) = X_N(\gamma(t)) \quad (12)$$

Where $X : \mathcal{S} \rightarrow T\mathcal{S}$ is called a vector field. $T\mathcal{S}$ is the tangent space of state space. A frame is precisely what allows us to define a basis to express any vectors of $T\mathcal{S}$, for

all points of \mathcal{S} . Let's try to express this particular X_N vector field. Newton's 2nd law states that $m\vec{a} = m\ddot{x} = \sum F(x, \dot{x}, t)$. The vector field associated to such equation is

$$X_N(x, \dot{x}) = \left(\dot{x}, \frac{1}{m} \sum F(x, \dot{x}, t) \right) \quad \text{NEWTON VECTOR FIELD} \quad (13)$$

Eqs. 12 is a natural rewriting of Newton equation in the language of differential geometry. It is called a dynamical system on state space and is generated by a vector field X , whose integral curves correspond to the trajectories of the system.

Let's finally define what type of objects is a physical quantity trajectory dependent. We call them "observable" (because we can get a value for any state of the system). An observable is therefore a function from state space to the real set:

$$\mathcal{O} : \mathcal{S} \rightarrow \mathbb{R} \quad \text{AN OBSERVABLE} \quad (14)$$

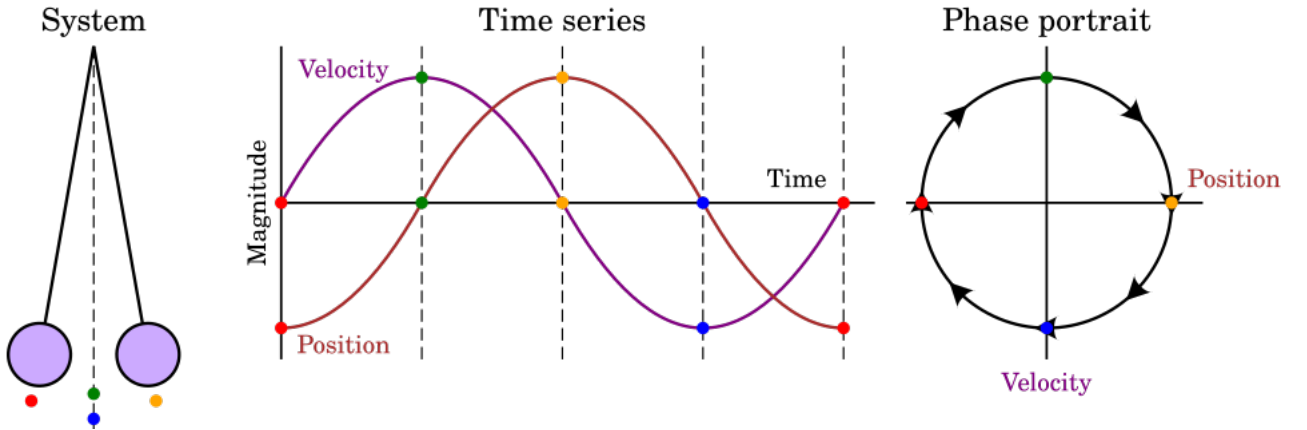


Figure 4: Illustration of conservation using the example of a pendulum. (Left): physical motion of the system. (Middle): position $x(t)$ and velocity \dot{x} as functions of time. (Right): representation of the motion in state space (x, \dot{x}) , also called phase space. Each point represents a state of the system. The trajectory traced in this space is generated by the Newton vector field X_N . For a conserved observable such as the mechanical energy, the trajectory remains confined to a curve where the observable has constant value.

Now that we introduced properly observable and made a clean distinction between a state and a trajectory, we can finally define conservation. An observable \mathcal{O} is a conserved quantity in physics if and only if

$$\forall \gamma : \mathbb{R} \rightarrow \mathcal{S} / \partial_t \gamma = X_N(x, \dot{x}), \frac{d}{dt} \mathcal{O}(\gamma(t)) = \mathcal{O}(x(t), \dot{x}(t)) = 0 \quad \text{CONSERVATION} \quad (15)$$

The condition $\partial_t \gamma(t) = X_N(x, \dot{x})$ is what enforce γ to be a physical trajectories, ie a solution of the equation of motion. A quantity \mathcal{O} is then conserved if its value remains constant along any physical trajectory.

Historically, conserved quantities were called "first integral" quantities, because they were constant after being integrated once with respect to time, whatever the physical path we choose. They are called *constants of the motion*.

Work is not a conserved quantity, because it is not an observable. Instead, it is defined for **one specific trajectory** $\gamma(t)$ along which we choose to integrate it: the one that transform the system from state A to state B. This is why we say that work is not a state function. Kinetic energy on the other hand, can be expressed as a function of a state, even though it depends only on \dot{x} and not on x .

$$E_c(s) = E_c(0, \dot{x}) = \frac{1}{2}m\dot{x}^2 \quad \text{but} \quad W(t) = \int_{\gamma(t)} \vec{F}(x, \dot{x}, t) \cdot d\vec{r} \neq W(x, \dot{x}) \quad (16)$$

We can explicit even further what this means for an observable (or state function) to be conserved. Let's apply the chain rule and express the total differential with respect to time:

$$\frac{d\vec{\mathcal{O}}}{dt} (\gamma(t), \dot{\gamma}(t), t) = \frac{\partial \vec{\mathcal{O}}}{\partial t} + \frac{\partial \gamma}{\partial t} \frac{\partial \vec{\mathcal{O}}}{\partial \gamma} + \frac{\partial \dot{\gamma}}{\partial t} \frac{\partial \vec{\mathcal{O}}}{\partial \dot{\gamma}} \quad (17)$$

If we assume $\vec{\mathcal{O}}$ depends on 3 coordinates, the above equation rewrites as:

$$\begin{aligned} \frac{d\vec{\mathcal{O}}}{dt} (\vec{\gamma}(t), \dot{\gamma}(t), t) &= \frac{\partial \vec{\mathcal{O}}}{\partial t} + \dot{\vec{\gamma}} \cdot \vec{\nabla}_{\vec{\gamma}} \vec{\mathcal{O}} + \ddot{\vec{\gamma}} \cdot \vec{\nabla}_{\dot{\gamma}} \vec{\mathcal{O}} \\ &= \partial_t F_i + \dot{\gamma}_j \partial_{\gamma_j} \mathcal{O}_i + \ddot{\gamma}_j \partial_{\dot{\gamma}_j} \mathcal{O}_i \end{aligned} \quad (18)$$

From now on we will use Einstein convention when writing equation using index notation: each repeated index is summed.

We can thus rewrite the condition of conservation of F from the equation above, as a "transport equation":

$$\partial_t \mathcal{O}_i + \dot{\gamma}_j \partial_{\gamma_j} \mathcal{O}_i + \ddot{\gamma}_j \partial_{\dot{\gamma}_j} \mathcal{O}_i = 0 \quad \text{TRANSPORT EQUATION} \quad (19)$$

This equation is only the consequence of "the observable \mathcal{O} is constant over all trajectories that are solutions of the equations of motion". This is why it is called a transport equation. It is not exactly the same as what physicist calls a conservation equation. Conservation equations are defined for extensive quantities, which do not exist in point mechanics. In physics we make a distinction between extensive and intensive quantities:

- An extensive quantity is a quantity volume dependent. As it depends on the size of the system, it can be written as the integral of a density. For instance, the mass is extensive, and we can write it as the integral of density over a volume.

- Meanwhile, an intensive quantity is a volume independent quantity. It cannot be written as the integral of a density over a volume, and won't change when the system size is modified.

Eq. 19 describes the conservation of an intensive observable called "tracer" along a trajectory defined by the flow X_N : we say that the tracer is advected by the trajectory, it follows it without changing. We will derive conservation equations in the chapter on continuum mechanics, after introducing extensive quantities.

Note that the following differential operator

$$\mathcal{L}_X = \dot{\gamma}_j \partial_{\gamma_j} + \ddot{\gamma}_j \partial_{\dot{\gamma}_j} \quad (20)$$

is called a Lie derivative along the vector field X (here $X = X_N$ as γ satisfies the equation of motion). There is a beautiful interpretation of this in the language of dynamical system and differential geometry, that we detail in appendix.

Now that we have properly defined the meaning of conservation in point mechanics, this leads to a natural question: are momentum and kinetic energy conserved quantity? Can we find other other interesting state functions that are conserved ?

4.1.3 Assumptions behind the conservation of momentum, mechanical energy and angular momentum

Momentum conservation

Let's start by checking if momentum is indeed an observable; and if therefore it can be an object that could possibly be defined as "conserved". Momentum at time t_0 only depends on velocity at t_0 , it is a function of the state $s = (0, \dot{x}(t_0))$, ie an observable.

$$p = m\dot{x}(t_0) \Rightarrow p = ms(t_0) \quad (21)$$

So momentum is an observable, it depends on a particular state $s_p = (0, \dot{x})$.

The key point to understand is that a momentum can be define for any possible trajectory we want $(\gamma(t), \dot{\gamma}(t))$: it does not depend on one in particular. Momentum is therefore an observable. When is it conserved?

Trivially, the momentum observable is conserved when its total time derivative vanishes for all trajectories solving the equation of motion. This is just Newton's laws ! Momentum is therefore conserved at equilibrium i.e. when the sum of forces is zero. It can also be conserved for an isolated system in a Galilean frame, where no forces is applied. let's time integrate Newton second law at equilibrium between t_0 and t_1 :

$$\frac{d\vec{p}}{dt} dt = 0 \Leftrightarrow \sum_{k \in M} \vec{F}_k = 0 \quad (22)$$

Where M denotes the set of forces applied on the considered system. Hence the momentum is always preserved at equilibrium.

Physical meaning: we can try to build an intuition on what this means. If we the forces applied on the system vanishes in a Galilean frame, by definition of such frames, the motion is rectilinear uniform. In other words, the system is *invariant by translation*: as the motion follows a line, nothing changes if we look at the trajectory at some point or another of the line it describes. We can for instance think of an asteroid far away from any celestial bodies in space: it is at equilibrium, and its motion just describes a motion. Nothing changes if we look at its motion from a position x_0 or from another x_1 . It just follows a line, as long as it is at equilibrium/isolated. In this situation, the asteroid momentum is conserved.

If the laws of physics are identical at every point in space (translation symmetry), then shifting the whole experiment by a constant vector cannot change the dynamics. This symmetry implies the existence of a conserved quantity: momentum.

Energy conservation

Now we can certainly ask the same question for kinetic energy. Work is not conserved because it is not a state function, but kinetic energy is. So when is it conserved? And more importantly, what does energy means?

Again, we start by writing down what it means for kinetic energy to be conserved, and integrate between t_1 and t_2

$$\frac{dE_c}{dt} = 0 \quad (23)$$

We now use the work-energy theorem:

$$\frac{dE_c}{dt} = \frac{d}{dt} \sum_k W_k \quad (24)$$

Conservation happens if and only if the time differential of an observable vanishes, but the time derivative of work (which isn't an observable) is not zero in general. Indeed,

$$W_k = \int_{x_A}^{x_B} \vec{F}_k \cdot \vec{dr} = \int_{t_A}^{t_B} \vec{F}_k \cdot \vec{v} dt \Rightarrow \frac{dW_k}{dt} = \vec{F}_k \cdot \vec{v} \neq 0 \quad (25)$$

The second integral is obtained by posing $\vec{dr} = \vec{v} dt$ in the first one.

However, it is possible to show the conservation of another quantity. Let's assume the considered forces can be written as the gradient of a scalar quantity we will call "potential":

$$\vec{F} = -\vec{\nabla} V(x, \dot{x}, t) \quad \text{POTENTIAL ENERGY} \quad (26)$$

Let's consider first its time derivative:

$$\begin{aligned}
\sum_k \frac{dV_k}{dt} &= \sum_k \left(\frac{\partial V_k}{\partial t} + \frac{\partial V_k}{\partial x_i} \frac{\partial x_i}{\partial t} + \frac{\partial V_k}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial t} \right) \\
&= \sum_k \left(\frac{\partial V_k}{\partial t} + \vec{\nabla} V_k \cdot \vec{v} + \vec{\nabla}_{\dot{x}} V_k \cdot \vec{a} \right) \\
&= \sum_k \left(\frac{\partial V_k}{\partial t} - \vec{F}_k \cdot \vec{v} + \vec{\nabla}_{\dot{x}} V_k \cdot \vec{a} \right) \\
&= \sum_k \left(\frac{\partial V_k}{\partial t} - \frac{d}{dt} W_k + \vec{\nabla}_{\dot{x}} V_k \cdot \vec{a} \right) \\
&= \sum_k \left(\frac{\partial V_k}{\partial t} + \vec{\nabla}_{\dot{x}} V_k \cdot \vec{a} \right) - \frac{dE_c}{dt}
\end{aligned} \tag{27}$$

If we then assume that all the potential involved V_k are **time and velocity independent**, $\partial_t V_k = \vec{\nabla}_{\dot{x}} V_k = 0$ and V_k is only a function of x so it is an observable (depend on state $s = (x, 0)$). We then arrive under these assumptions at

$$\frac{dE_c}{dt} + \sum_k \frac{dV_k}{dt} = 0 \tag{28}$$

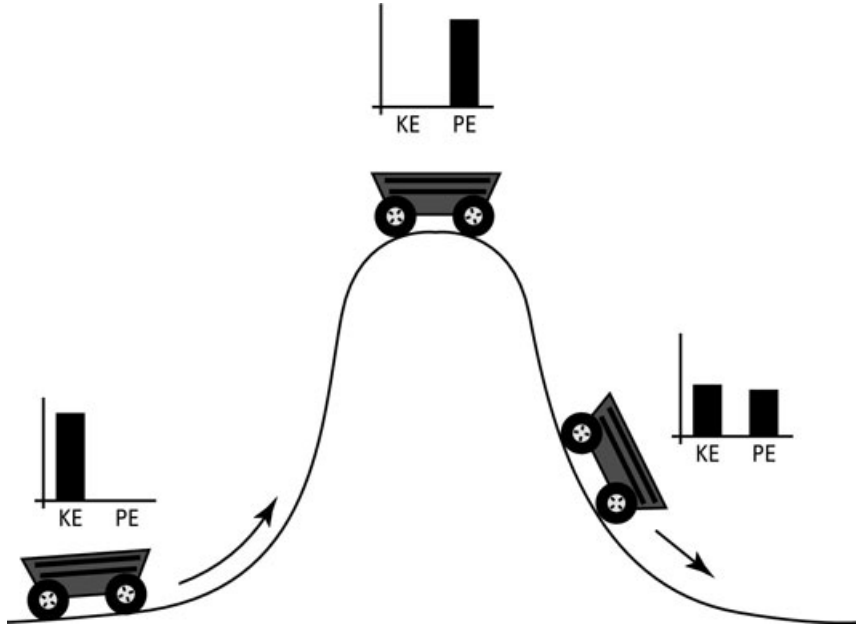


Figure 5: Illustration of the conservation of mechanical energy: kinetic energy alone cannot be conserved in general, as any velocity variation would modify its value with respect to time. This is why potential energy is a convenient way of representing energy conservation: when velocity decreases, the potential energy increases. This is however coming from some assumptions on the force: they must be conservative, and the potential must be time and velocity independent. Image taken from ref. [6]

In conclusion, the conserved observable is almost never E_c (except of $\vec{F}_k \cdot \vec{v} = 0$

which is typically the case of the Lorentz force). It is a quantity called mechanical energy:

$$E_m = E_c + \sum_k V_k \quad \text{MECHANICAL ENERGY} \quad (29)$$

We remind that this quantity is conserved only under two assumptions:

- The considered forces on the system can be written as the gradient of a scalar potential. Such forces are thereby called **conservative forces**.
- In addition, the scalar potential must be an observable (otherwise we cannot call it a conserved quantity, and the partial derivative with respect to time is not zero).
- Finally, the scalar potential must only depend on position and not on velocity (otherwise the $\nabla_{\dot{x}} V_k \cdot \vec{a}$ term must be kept in Eq. 28 and mechanical energy is not conserved anymore). In such situations the conserved energy must be defined differently, which requires the more general framework of Lagrangian mechanics.

Later we will show with Noether theorem how any form of conservation is equivalent to a symmetry of the system. We can for now already get a feeling of this: let's try to find the conserved quantities in point mechanics, and get an intuition of what this could mean in terms of symmetries.

A deeper interpretation of this result comes from the symmetry of the equations of motion. When the potential does not depend explicitly on time $V = V(x)$, the force acting on the system is also independent of time. The equations governing the motion therefore remain unchanged if the entire experiment is shifted by a constant amount of time $t \rightarrow t + \tau$. In other words, the laws of physics do not depend on the instant at which the experiment is performed. This property is called invariance under time translation. When such a symmetry holds, the dynamics admits a conserved observable. In the present case, this conserved quantity is the mechanical energy $E_m = E_c + \sum_k V_k$. Energy conservation therefore expresses the fact that the laws of mechanics are the same at all moments in time. Note that in the more general framework of Euler-Lagrange equations, the potential can depend both on position and velocity. In that case, the conserved energy is not just kinetic plus potential energy; but the time invariance interpretation remains the same.

The modern concept of energy emerged gradually during the eighteenth and nineteenth centuries. Leibniz vis viva was not yet interpreted as a universal conserved quantity, it anticipated the modern kinetic energy. During the nineteenth century, a series of experimental and theoretical developments revealed that mechanical work, heat, and other physical processes were different manifestations of a single conserved

quantity. This insight was established through the work of scientists such as James Prescott Joule, Julius Robert von Mayer, and Hermann von Helmholtz, who formulated the general principle of conservation of energy. Much later, the deep origin of this conservation law was clarified mathematically by Emmy Noether, who showed that energy conservation follows from the invariance of the laws of physics under translations in time.

Angular momentum

Angular momentum was noticed by early thinkers like Archimedes by studying the effect of forces on levers and rotations, and implicitly considered by Newton. However, it was really first defined in the XVIIIth century by Euler and D'Alembert, as

$$\boxed{\vec{L} = \vec{r} \wedge \vec{p}} \quad \text{ANGULAR MOMENTUM} \quad (30)$$

As usual, we start by considering the time derivative of the angular momentum:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \wedge \vec{p} + \vec{r} \wedge \frac{d\vec{p}}{dt} \quad (31)$$

But

$$\frac{d\vec{r}}{dt} \wedge \vec{p} = m \frac{d\vec{r}}{dt} \wedge \frac{d\vec{r}}{dt} = \vec{0} \quad (32)$$

And, using Newton 2nd law we get:

$$\boxed{\frac{d\vec{L}}{dt} = \vec{r} \wedge \frac{d\vec{p}}{dt} = \vec{\mathcal{T}}} \quad \text{ANGULAR MOMENTUM THEOREM} \quad (33)$$

$\vec{\mathcal{T}}$ is called the torque (or moment) of the system, and can be expressed as a function of the applied force on the system using Newton 2nd law:

$$\vec{\mathcal{T}} = \vec{r} \wedge \frac{d\vec{p}}{dt} = \vec{r} \wedge \sum_k \vec{F}_k = \sum_k \tau_k \quad (34)$$

τ_k is the torque or moment associated to a given force.

Trivially, if the sum of all the torque vanishes, we get the conservation of angular momentum. This typically happens for **central forces**, that are proportional to the position vector (i.e. the distance from the origin). More generally, the angular momentum is conserved if the torque of the system vanishes: this only requires that the sum of applied forces is proportional to the position vector:

$$\vec{\mathcal{T}} = \vec{r} \wedge \frac{d\vec{p}}{dt} = \vec{0} \Leftrightarrow \frac{d\vec{p}}{dt} = \sum_k \vec{F}_k \propto \vec{r} \quad (35)$$

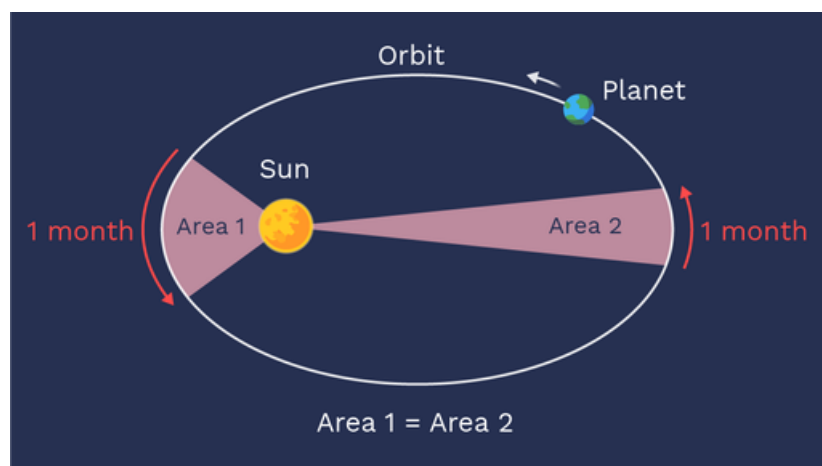


Figure 6: Kepler law is a consequence from angular momentum conservation. Indeed, for astral bodies, only the gravitational force (which is a central force) is applied. As a consequence from this conservation, the radius vector from the Sun to the planet sweeps out equal areas in equal times. This is **Kepler's second law**. Taken from [here](#)

If the resultant force acting on the system is always parallel to the position vector, it depends only on the distance from the origin. Such forces are called central forces. Since a rotation of the coordinate system does not change this distance, the force remains unchanged under rotations. The equations of motion are therefore invariant under rotations of space. This rotational symmetry is associated with the conservation of angular momentum.

During the XIXth century, the rotational interpretation of angular momentum was provided by Poinsot, while Hamilton and Lagrange showed the consequence of rotational symmetry on momentum. Kepler's second law (planets sweep out equal areas in equal times) were reinterpreted as a consequence of angular momentum conservation. **Kepler's second law** (planets sweep out equal areas in equal times) was later understood as a direct consequence of angular momentum conservation in the gravitational field.

4.1.4 Classical harmonic oscillators and waves

Soon