8 Electromagnetism

8.1 Introduction to electromagnetism

8.1.1 A brief story of the electric charges

Electricity

The very first observation of an electrostatic phenomenon dates back to the ancient Greeks. Around 600 BCE, Thales reported rubbed amber attracted light objects like straw. In 1600, William Gilbert published "De Magnete" where he introduced the term "electricus" and distinguished magnetism from electrification. He showed that many material exhibits electric attraction when rubbed, introducing the very first triboelectric scale.

In the XVIIIth century, the word "charge" emerges. Charles du Fay distinguished two types of electricity (vitreous and resinous) depending on whether they attract or repel objects. This was the early day of electric charge. Benjamin Franklin suggested a conserved "electric fluid", whose excess or deficit correspond to positive or negative charge. This was one of the earliest formulation of charge conservation. In 1752, his kite experiment proved thunder was an electric phenomenon. In 1785, Coulomb used a torsion balance to measure electric forces. He showed

$$F_c = k\frac{q_1 q_2}{r^2} \tag{31}$$

Where k is a constant, q_1 and q_2 the total charge of two systems, standing at a distance r from each other. During the XIXth century, we could cite the first batteries by Volta as well as Faraday's work on electrolysis. Indeed, Faraday showed that the amount of chemical reaction was proportional to the current, pinpointing that charges are discrete and associated with microscopic phenomenons.

In 1897, Thomson work on Cathodic rays (see Sec. 12.1.1) identified a universal negatively charged particle, the electron.

The elementary charge: Millikan experiment

We could finally describe the Millikan experiment in 1909, that provided a precise measurement of the elementary charge.

In this experiment, some oil drops go through a hole. The drops are ionized by X-rays, and therefore acquire a charge. Under the action of gravity and the friction of the fluid, they reach a terminal velocity v_1 given by expressing the fluid force with Stokes law (assuming the oil drops are perfect spheres):

$$F_f = 6\pi r\eta v_1 \tag{32}$$

As the oil drop reaches a terminal velocity, F_f and the action of gravity F_g on the drop are equal:

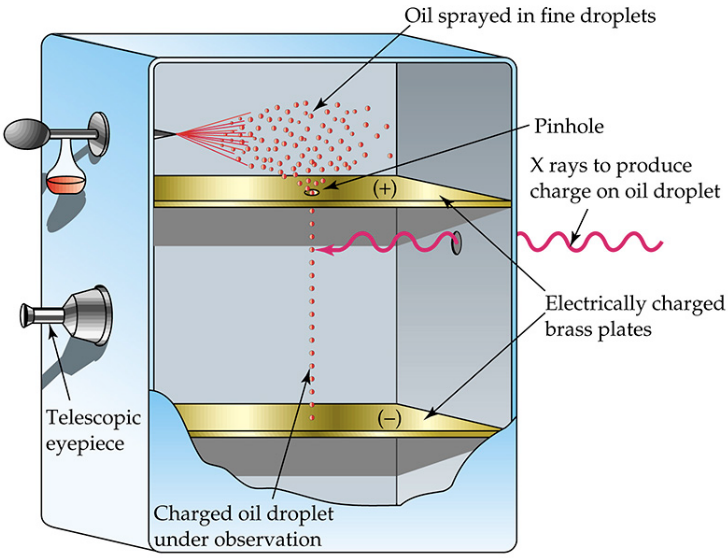

Figure 4: The Millikan (or oil drop) experiment. To evaluate the elementary charge, some ionised oil drops are send through a cylinder. The oil drops reach under gravity a terminal velocity because of air friction. An electric field is then turned on and adjusted to reach equilibrium, allowing to express the value of the elementary charge e.

$$F_g = F_f = \frac{4}{3}\pi r^3(\rho - \rho_{air})g \tag{33}$$

In that expression, the effective weight is reduced by buoyancy. According to Archimedes' principle, the upward force equals the weight of the displaced air, which leads to replacing ρ by $\rho - \rho_{air}$. From that equilibrium equation we can extract the radius of the oil drop r. If we now switch on an electric field $\overrightarrow{E}$ (with $\overrightarrow{E} = \frac{V}{d}$ for 2 parallel plates with a potential difference V and at distance d), the particle will rise and reach another terminal velocity v_2 (again because of the fluid viscosity and using Stokes law). In this case, the equilibrium is now between weight, electric force and friction:

$$F_e + F_g + F_f = 0 \tag{34}$$

And as r is known, F_g and $F_f = 6\pi r\eta v_2$ are known. Therefore, since the electric force is given by $F_e = qE$ and $E = \frac{V}{d}$ is also known, we can isolate q, the charge of the oil drop:

$$q = -\frac{1}{E}\left(\frac{4}{3}\pi r^3(\rho - \rho_{air})g + 6\pi r\eta v_2\right) \tag{35}$$

Once the charge of one oil drop is known, the experiment is repeated multiple time. The charge of all the drops is found to be multiples of an elementary charge $e = -1,602.10^{-19}C$. While the Millikan experiment itself had some controversy and isn't the most accurate experiment (assumptions: Cunningham correction is ignored (see Eq. 27 in Sec. 5.3), low Reynolds fluid, spherical drop), it remains today one of the most pedagogical setup to assess the elementary charge e.

Charge conservation and densities

The oil drop experiment is one among many experiments proving that all electric charge of atoms and molecules are multiple of the elementary charge. As we are dealing with discrete objects, we can apply the same reasoning than in Sec. 5 and define a density of charge ρ such that

$$q = \int_{\Omega} \rho(\overrightarrow{r}, t) d^3\overrightarrow{r} \tag{36}$$

If we now ask the question "how much charge Δq go through a wire during Δt, we arrive naturally at the usual definition of the electric current intensity $I = \frac{\Delta q}{\Delta t}$. We therefore define the electric current density $\overrightarrow{J}$ such that

$$I = \int_{\partial\Omega} \overrightarrow{J} \cdot \overrightarrow{n} d^S \tag{37}$$

This definition is exactly the same as in continuum mechanics with mass and density. The current density simply answers "how much current in this region of space", while the electric current intensity I is a global result on an area. We could also define it from the microscopic current density in the thermodynamic limit. Indeed, for a discrete set of charges we have:

$$\overrightarrow{J}(\overrightarrow{r}, t) = \sum_i q_i(t) v_i(t) \delta(\overrightarrow{r} - \overrightarrow{r}_i(t)) \tag{38}$$

In the thermodynamic (or coarse-grained) limit, assuming that a local average velocity field can be defined over a small volume containing many charges, the microscopic current density reduces to

$$\overrightarrow{J}(\overrightarrow{r}, t) = \rho(\overrightarrow{r}, t)\overrightarrow{v}(\overrightarrow{r}, t) \tag{39}$$

It is interesting to note that these two constructions of charge and current densities originate from opposite viewpoints. The macroscopic definition starts from a finite control volume Ω and defines densities through fluxes and balance laws. In contrast, the microscopic approach begins with discrete particles and defines densities through distributional sums, which are then coarse-grained over a small volume containing many charges. This coarse-graining volume is taken small compared to macroscopic scales, yet large enough to contain a large number of particles, ensuring smooth fields (later we will see that it is a very general aspect of physics: macroscopic densities are defined using field, but microscopic densities comes from measures and distributions).

To conclude this section, we have everything we need to write down the continuity equation of charge. It is very similar to the one defined with mass in continuum mechanics:

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{J} = 0 \quad (40)$$

As we already mentioned, this is the typical form of a balance equation: the infinitesimal variation of a quantity at time t (here the charge density ρ) is given by how much charges enter or leave the domain at that time t. In other word, by the divergence of the electric current density. If it is positive, if we consider an infinitesimal variation around t, charges leave the domain. Conversely if it is negative, charges enter the domain. Let's finally note that if we were to have a source or sink that create or destroy charge in the domain $S(\overrightarrow{r}, t)$, the right hand side would be equal to that term instead of 0 (variation of the quantity = what enter or exit + what's created or destroyed inside).

Magnetism

Like electricity, magnetism was observed by the ancients, but its experimental history is different. Natural magnets (or lodestones) were known to attract iron, and their directional behavior gave rise to the first compasses for navigation.The word magnetism comes from magnesia, the name of an antic city were ferromagnetic pebbles where observed (exact details of how magnetism have been first observed and described by ancients is a research topic, the interested reader may refer to [5]). William Gilbert, in the early 1600s, distinguished magnetic phenomena from electrification, showing that magnetism is an intrinsic property of certain materials rather than a consequence of rubbing or friction. Unlike electric charges, which can exist isolated as positive or negative, magnetic poles are always found in pairs: cutting a magnet in half simply produces two smaller magnets, each with a north and south pole.

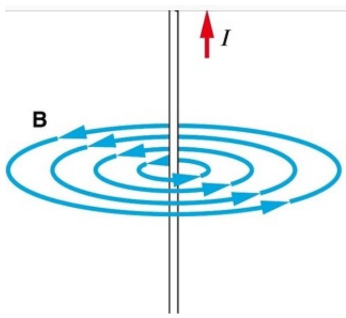

Figure 5: Orsted observed a current going through a wire generate around the wire a magnetic field forming loops of vanishing intensity, centered around the wire

In 1819, there were many experiments showing a correlation between electricity and magnetism already. That particular year, Oersted observed for instance that electric current moving through a wire, the needle of a compass started to move, creating the first prototype of a galvanometer to measure current. Mathematically, the magnetic field $\overrightarrow{B}$ caused by a current distribution $\overrightarrow{J}$ is given by the Biot-Savart law:

$$\boxed{\overrightarrow{B}(\overrightarrow{r}) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{\overrightarrow{J}(\overrightarrow{r'}) \times (\overrightarrow{r} - \overrightarrow{r'})}{|\overrightarrow{r} - \overrightarrow{r'}|} d^3\overrightarrow{r}} \qquad \textbf{BIOT-SAVART LAW} \qquad (41)$$

where μ_0 is the permeability of free space and will be discussed later. This law really captures the fundamental observation that magnetic fields circulate around currents. Magnetism is intimately tied to the motion of charges.

In 1821, Ampere published a first attempt to describe the coupling between magnetism and electricity with his famous law. It is in this context that Maxwell started his education first with his family, then with a preceptor. At only 14 he was already publishing a first paper on geometry !

8.1.2 Maxwell equations

From 1831 to 1855, Maxwell published more than 16 thousand entries in his laboratory, creating the first complete electromagnetic formalism [2].

Maxwell's equations formalize the interplay of electric and magnetic fields. They extend Coulomb's and Ampère's law by including the displacement current, ensuring local charge conservation, and predicting that electric and magnetic disturbances propagate as waves at a finite speed—identified with the speed of light. These four equations therefore unify previously disparate phenomena: electrostatics, magnetostatics, and electrodynamics. His famous 4 main equations written on his grave are the following:

$$\boxed{\begin{array}{rclcrcl} \nabla \cdot \mathbf{E} & = & \dfrac{\rho}{\epsilon_0} \ \text{(A)} & & \nabla \cdot \mathbf{B} & = & 0 \ \text{(B)} \\ \nabla \times \mathbf{E} & = & -\dfrac{\partial \mathbf{B}}{\partial t} \ \text{(C)} & & \nabla \times \mathbf{B} & = & \mu_0 \mathbf{J} + \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} \ \text{(D)} \end{array}} \qquad (42)$$

With (A): Maxwell-Gauss, (B): Maxwell-Flux, (C): Maxwell-Faraday and (D): Maxwell-Ampere.

Maxwell-Gauss

Equation (A), Maxwell-Gauss, takes its name from a contribution of Gauss in the 1800s: indeed, if one integrate Coulomb force (Eq. 31) on a sphere $\mathcal{S}$, on gets a constant:

$$\oint_{\mathcal{S}} \overrightarrow{E}(r) \cdot d^2\overrightarrow{S} = \oint_{\mathcal{S}} \frac{q\overrightarrow{r}}{k|r^3|} \cdot \overrightarrow{r}|r|sin(\theta)d\theta d\phi = 4\pi kq \qquad (43)$$

Later, physicists homogenized these equations and ended up posing:

$$k = \frac{1}{4\pi\epsilon_0} \tag{44}$$

Back to Gauss computation, it is clear that using his famous divergence theorem leads to

$$\int_{\mathcal{S}} \overrightarrow{\nabla} \cdot \overrightarrow{E} dV = \frac{q}{4\pi\epsilon_0} \tag{45}$$

Hence, by introducing the charge density as in Eq. 36, and using the fact the integration can be done on any arbitrary volume, Maxwell ended up to this strong form condition on the divergence of the electric field.

Maxwell-Gauss can therefore just be seen as the generalization in the continuum assumption of Coulomb law.

Maxwell-Flux

Equation (B), Maxwell flux, is a direct consequence of the Biot-Savart law (Eq. 41) expressing the simple observation of a magnetic flux generated by a current. Indeed, let's compute the divergence of the law:

$$\overrightarrow{\nabla} \cdot \overrightarrow{B}(\overrightarrow{r}) = \frac{\mu_0}{4\pi} \int_{\omega} \frac{\overrightarrow{\nabla}_r \cdot \overrightarrow{J}(\overrightarrow{r'}) \times (\overrightarrow{r} - \overrightarrow{r'})}{|\overrightarrow{r} - \overrightarrow{r'}|^3} d^3\overrightarrow{r'} \tag{46}$$

Where $\overrightarrow{\nabla_r}$ indicates we derivate with respect to the position vector of the magnetic field and not the integration variables. We can then use the differential geometry identity:

$$\overrightarrow{\nabla} \cdot (\overrightarrow{A} \times \overrightarrow{C}) = \overrightarrow{C} \cdot (\overrightarrow{\nabla} \times \overrightarrow{C}) - \overrightarrow{A} \cdot (\overrightarrow{\nabla} \times \overrightarrow{C}) \tag{47}$$

with $\overrightarrow{A} = J(r')$ which is a constant if derivated with respect to r and therefore has a vanishing curl, and $\overrightarrow{C} = \frac{\overrightarrow{r} - \overrightarrow{r'}}{|\overrightarrow{r} - \overrightarrow{r'}|^3}$. $\overrightarrow{C}$ is a radial function (it can be wrote as $F(\overrightarrow{r})\frac{\overrightarrow{r}}{|\overrightarrow{r}|}$, here with $F(\overrightarrow{r}) = \frac{1}{|\overrightarrow{r} - \overrightarrow{r'}|^3}$). For such functions, the curl is zero everywhere except at the origin. For $\overrightarrow{C}$, this means the curl vanishes everywhere but when $\overrightarrow{r} = \overrightarrow{r'}$. Such a condition means we are trying to consider the magnetic field inside the wire. Biot-Savart law is only defined outside the current distribution. Hence, assuming we are considering only the magnetic field generated outside of a current distribution, the curl of $\overrightarrow{C}$ is also zero. This thus lead to the Maxwell-Flux law:

$$\nabla \cdot \overrightarrow{B} = 0 \tag{48}$$

Integrated over a domain, this law also means that the magnetic flux is conservative. Indeed:

$$\int \nabla \cdot \overrightarrow{B} d\Omega = \oint_{\partial\Omega} \overrightarrow{B} \cdot \overrightarrow{n} dS = 0 \tag{49}$$

In other word, as in fluid mechanics with uncompressible flows, magnetic fluxes are always conserved: on the surface of a given arbitrary volume, the incoming flux is always the opposite of the outcoming flux.

Maxwell-Faraday

Maxwell-Faraday really is the law that we are using when we want to generate electricity or magnetism. Discovered in 1831 by Faraday the law represent an experimental phenomenon called "induction".

- A magnet moved near a coil generate a current.
- A coil moved near a magnet generate a current.
- Changing the magnetic field inside a coil generate a current.

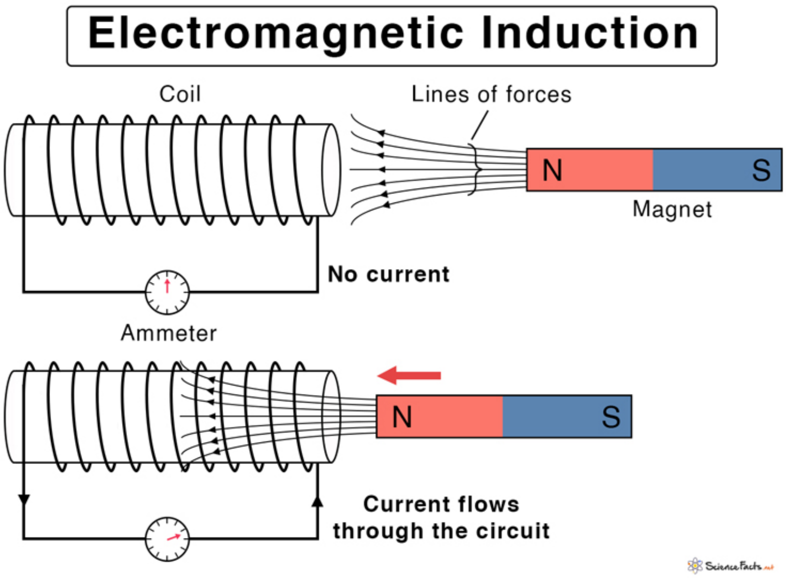

Figure 6: If the magnet is not moving, the emitted magnetic field does not change with time. No current is created. If we move the magnet, the time variation will however create a current. This is induction. The same observation can be done using a magnet that emits a field significantly changing with temperature: without even moving the magnet, if we cool or heat it, the time variation of the field will induce a current. Sketch taken from Science Fact.

These observations acted in a sense as the reciprocal of Biot-Savart law: if a magnetic field is generated from a current through a wire, a current is also induced in a wire from a magnetic field **variation**. Such a variation is observed when moving a magnetic, or by tuning a magnetic field next to a coil.

Later in 1834, Lenz observed the direction of the induced current is the opposite of the time variation of the magnetic field. But Maxwell realised that the phenomenon is not about the current, which is a material property. It's about the field itself.

At this stage, we also need to introduce Ohm's law, which translate how the electric field is related to the current:

$$\boxed{\vec{E} = \sigma \vec{J}} \qquad \textbf{OHM'S LAW} \tag{50}$$

Where σ is the isotropic electric conductivity (if it is anistropic, it becomes a tensor to describe directionality). This law simply says that the electric field is proportional to the current. The electric resistance R is often defined as the inverse of the conductivity. This law is an extension of the famous $U = RI$ law (with U the potential difference and I the electric current intensity). Indeed, for a wire of section A and length L, the current intensity is obtained by integrating the density: $I = |\vec{J}|A$. Similarly, the potential difference U on the wire is $U = |\vec{E}|L$. This thus leads to

$$\frac{I}{A} = \sigma \frac{U}{L} \tag{51}$$

or equivalently, $U = \frac{A}{\sigma L} I$. Defining the lineic electric resistance as $R = \frac{A}{\sigma L}$, we get back the well known $U = RI$.

To get back to Maxwell-Faraday, Maxwell realized that Faraday and Lenz observation is a material dependant properties. They showed that for all materials,

$$\sigma \vec{J} = -\frac{\partial}{\partial t} \vec{B} \tag{52}$$

but did not notice how this induced current by the field variation changed with the material of the wire. Maxwell on the other hand, generalized Ohm law (so the opposite of what we did!) and expressed that result using only fields, leading to a material independant law:

$$\vec{E} = -\partial_t \vec{B} \tag{53}$$

To summarize, the heart of this law is really induction, and the beautiful coupling between electric and magnetic field comes from Maxwell idea of material independent law, leading to a generalization of Ohm law.

Maxwell-Ampère

Taking inspiration from the Orsted wire experiment and the Biot-Savart law, in 1821 Ampère tried to formalize the interaction between electric and magnetic field. As magnetic field forms loops around the wire, it is only natural to express this dependency with a curl. To see it, we can start from the integral form: consider a closed curve forming a loop around a wire of section $\mathcal{S}$ crossed by an electric current density $\vec{J}$. Ampère law basically translates as the following:

$$\oint_{\mathcal{C}} \vec{B} \cdot \vec{dl} \propto \int\int_{\mathcal{S}} \vec{J} \cdot \vec{dS} = I_{tot} \tag{54}$$

In the above equation, I_{tot} is the total current intensity (integrated over the wire section, only place where it is non-zero). $\overrightarrow{dl}$ is a 1d integration variable tangent to $\mathcal{C}$. The law thus states that if we sum the intensity of the magnetic field on a loop around a wire, it is always proportional to the current intensity. Of course, the larger the loop, the weaker the magnetic field on a section of the loop, as the integral must be equal to a constant. So the magnetic field intensity also decreases as we move away from the wire. The proportionality constant is called μ_0 and is called the vacuum permeability. To get to the curl form, we can use Stokes theorem:

$$\oint_{\mathcal{C}} \overrightarrow{B} \cdot \overrightarrow{dl} = \int\int_{\mathcal{S}} \nabla \times \overrightarrow{B} dS \tag{55}$$

As we get an equality between two surface integral true for any surface $\mathcal{S}$, we get the strong form of Ampère's law:

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \mu_0 \overrightarrow{J} \tag{56}$$

Later, Maxwell saw a paradox in the current state of this law. If one consider a charging capacitor (two parallel metallic plates separated by an insulator), it emits a magnetic field even though no current flows through the insulator. Using Ampère's law would give zero curl in the gap, which is in contradiction with observations (see Fig. 7).

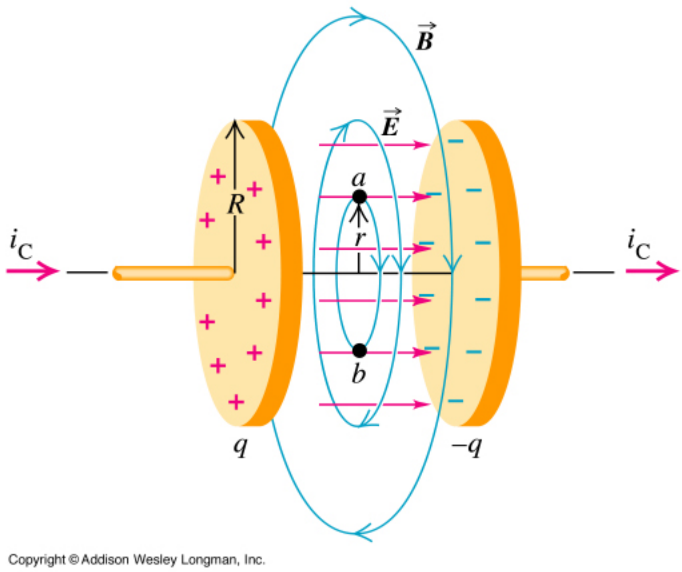

Figure 7: Representation of the electric (red lines) and magnetic (blue lines) on a capacitor. Taken from here

In addition, charge conservation requires $\overrightarrow{\nabla} \cdot \overrightarrow{J} = \frac{\partial \rho}{\partial t}$, which is violated if we take the divergence of Ampère's law. For these reasons, Maxwell introduced a "displacement current" $\overrightarrow{J_d} = \epsilon_0 \frac{\partial \overrightarrow{E}}{\partial t}$. This is not a physical current, it does not represent the displacement of real charges. It should rather be seen as a term accounting for electric field variations and imposed to verify charge conservation. Hence we arrived at the final version of Maxwell-Ampère equation:

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \mu_0 \left(\overrightarrow{J} + \epsilon_0 \frac{\partial \overrightarrow{E}}{\partial t} \right) \tag{57}$$